A trader is a person or entity, in finance, which buys and sells financial instruments such as stocks, bonds, commodities, derivatives, and mutual funds in the capacity of agent, hedger, arbitrageur, or speculator. Traders buy and sell financial instruments traded in the stock markets, derivatives markets and commodity markets, comprising the stock exchanges, derivatives exchanges, and the commodities exchanges. Several categories and designations for diverse kinds of traders are found in finance, these may include: local, floor, high-frequency, pattern day, rogue and stock traders. All types of these market participants need know when enter and when exit of a financial position. This question is answered using the optimal stopping techniques. In mathematics, the theory of optimal stopping or early stopping is concerned with the problem of choosing a time to take a particular action, in order to maximize an expected reward or minimize an expected cost. Optimal stopping problems can be found in areas of statistics, economics, and mathematical finance (related to the pricing of American options). A key example of an optimal stopping problem is the secretary problem. Optimal stopping problems can often be written in the form of a Bellman equation, and are therefore often solved using dynamic programming, see Tijms (2012) and references therein.

This paper develops some results about the optimal exit or enters to a position in a financial market. To this end, let $s_t$ be the price of a financial asset like stock price at time $t=1,\ldots,n$ and $r_t = \frac{s_t-s_{t-1}}{s_{t-1}}$ be the related return process. Suppose that conditional on some processes such as $\mu_t$, then $r_t$'s are independent random variables with mean $\mu_t$. Indeed, $r_t = \mu_t + \sigma z_t$ where $z_t$'s are independent standard normal $N(0,1)$ distributed random variables. Also, assume that $\mu_t$'s are independent and also mutually independent of $z_t$'s with common distribution $N(\gamma,\nu^2)$. Conditional on $\mu_t$'s, it is interested to find the stopping time $\tau$ which maximizes $E(s_\tau)$.

To this end, using the dynamic programming solution of above mentioned optimal stopping problem, it is seen that

$$\pi_t = \max(s_t, E(\pi_{t+1}|F_t)), t = 1,2,\ldots,n-1,$$

where $F_t$ is the $\sigma$-field generated by $s_i, i=1,2,\ldots,t$. Notice that $\pi_n = s_n$. Using the Markov property of $s_t$, it is seen that $E(\pi_{t+1}|F_t) = E(\pi_{t+1}|s_t)$. By recursive solution of above equation, it is seen that $\pi_{n-1} = (1+\gamma)n \pi_{n-1}, \pi_{n-2} = (1+\gamma)n \pi_{n-2}, \ldots$ at which

$$\begin{align*}
Y_n &= (1 + \mu_n) - 1 = \mu_n \\
Y_{n-1} &= (1 + \mu_{n-1})(1 + \gamma n) - 1 \\
Y_{n-2} &= (1 + \mu_{n-2})(1 + \gamma n) - 1 \\
\vdots
\end{align*}$$

Here, $x^* = \max(x,0)$. Indeed, one can see that $\pi_{n-1} = (1+\gamma)n \pi_{n-1}$ where $\gamma = (1+\mu_n)(1+\gamma)n - 1$ with initial value $\gamma = \mu_n$. The stopping time $\tau$ is the random time that $\pi_{n-\tau} = s_{n-\tau}$. Equivalently, $\gamma = 0$. Let $\tau_n \geq \tau_{n+1} \geq \cdots$ be the possible values of $\tau$ and let $\lambda_{n,j} = \tau_{n+1}, \tau_{n+2}, \cdots \geq 1$ and $\tau_n = n$. Clearly, $\lambda_n$'s are independent.

The rest of paper is designed as follows. In the next section, two real data sets are analyzed by supposing a noisy step function for mean function. Then, alternative ways for formulation of time varying mean functions are proposed. Section 3 concludes.
EMPIRICAL RESULTS

In this section, empirical results of above mentioned theoretical results are surveyed. This section has two parts. First, considering mean function of Yao (1984), two real data sets are analyzed. In the second sub-section, via Monte Carlo simulations, some alternative methods for modeling time varying mean functions are studied.

Real Data Sets

Here, two real data sets are analyzed. Indeed, a backward version of noisy discrete time step function of Yao (1984) is considered for modeling \( \mu_{n-i} \)’s as follows, in both two real data sets:

\[
\mu_{n-i} = (1 - J_{n-i}) \mu_{n-i+1} + J_{n-i} \zeta_{n-i},
\]

where \( \mu_i \) is kept fixed and \( J_i \)’s are independent and identically distributed Bernoulli random variables at which \( J_i = 1 \) if \( \mu_i - \mu_{i-1} \neq 0 \) with probability of \( p \) and the magnitude of changes are \( \zeta_i \) independent of \( J_i \) and have common distribution \((\delta, v^2)\).

(a) Apple Stock. The first data set contains the daily stock price of Apple Inc. for period of 7 Aug. 2018 to 28 Feb. 2020 including 393 observations. The rolling means (across a window of length 10) are derived and its plot is given as follows. Clearly, a non-stationary pattern in mean is seen. If \( \lambda > 0.001 \), it is assumed that \( J_i = 1 \) and \( \hat{\lambda}_i = \hat{\mu}_i - \hat{\mu}_{i-1} \). It is seen that \( p = 0.644, \delta = 5.023 \times 10^{-5}, v = 0.00321 \). Clearly, the normality of \( \hat{Z}_i \) is satisfied.

(b) Exchange rate. The second real data set contains the daily historical exchange rates for study period 5 Mar. 2018 to 3 Mar. 2020 involving 521 observations. The rolling means (across a window with length 10) are plotted as follows. Clearly, a step function is fitted to the mean function of this process. It is seen that \( p = 0.796 \) and \( \zeta_i \)’s are normally distributed with mean \(-0.00014\) and standard deviation \(0.001858\).

Alternative time varying \( \mu \)’s

Here, by running some simulations, alternative ways for modeling time varying \( \mu \)’s are surveyed. These ways contain independent \( \mu \)’s, random walk mean function, Bayesian setting, and finally CAPM modeling of time varying \( \mu \)’s. All formulations propose random mean function except, the last one.

Example 1 (Independent means). Assuming \( \mu_{i,j} = 1, 2, \ldots, n = 100 \) are independent random variables come from normal distribution with zero mean and standard deviation 0.1, the following plot gives the time series plot of \( \pi_{t-s}^\mu, t=1, \ldots, 100 \). It is seen that at \( \tau = 82, 93, 95, 96, 97, 98, 100 \) the plot is zero and they are optimal time for trading.

The following Table gives the mean and standard deviation of sampling first, second and third quintiles of stopping times at which \( \lambda = \mu \), over the sample size, i.e., \( \tau_{n-1}/n \). This Table gives useful information for traders.

<table>
<thead>
<tr>
<th>Quartiles</th>
<th>Mean</th>
<th>Stdev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td>0.397</td>
<td>0.238</td>
</tr>
<tr>
<td>Q2</td>
<td>0.629</td>
<td>0.197</td>
</tr>
<tr>
<td>Q3</td>
<td>0.801</td>
<td>0.165</td>
</tr>
</tbody>
</table>
Example 2 (Random walk mean process). In this simulated example, it is assumed that time varying process $\mu_t$ has a backward random walk structure, i.e.,

$$\mu_{n-t} = \mu_{n-t+1} + \zeta_{n-t}, n = 100$$

where $\zeta_n$'s are mutually independent and have common distribution normal distribution with zero mean and standard deviation $\beta$. Assume that $\beta=0.1$. Then, the following table gives the numbers (N) of matching $\mu$’s and $\gamma$’s.

<table>
<thead>
<tr>
<th>$\text{Mean}$</th>
<th>$\text{Stdev.}$</th>
<th>$\text{Q1}$</th>
<th>$\text{Q2}$</th>
<th>$\text{Q3}$</th>
<th>$\text{Skew}$</th>
<th>$\text{Kurt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>44.14</td>
<td>37.3</td>
<td>4</td>
<td>38.5</td>
<td>79.25</td>
<td>0.187</td>
<td>1.36</td>
</tr>
</tbody>
</table>

Next, consider a backward stationary first order auto-regressive AR(1) process for $\mu_{n-t}$ assuming that $\mu_{n-t} = \alpha \mu_{n-t+1} + \zeta_{n-t}$ where $\alpha=0.2$. Then,

Table 4. Summary statistics of (AR(1))

<table>
<thead>
<tr>
<th>$\text{Mean}$</th>
<th>$\text{Stdev.}$</th>
<th>$\text{Q1}$</th>
<th>$\text{Q2}$</th>
<th>$\text{Q3}$</th>
<th>$\text{Skew}$</th>
<th>$\text{Kurt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.3</td>
<td>9.07</td>
<td>8</td>
<td>15</td>
<td>22</td>
<td>0.42</td>
<td>2.43</td>
</tr>
</tbody>
</table>

Example 3 (Bayesian setting). Here, we adopt a Bayesian time varying structure for finding $\mu_t$. Suppose that $r_t$ has normal distribution with mean $\mu$ and constant (for simplicity arguments) volatility $\sigma$ (it could be supposed that volatility process obeys a heteroscedasticity process such as ARCH or GARCH series, which is omitted, here). Also, for simplicity reasons, consider a conjugate prior normal distribution for $\mu$, with hyper-parameters mean $\alpha$ and standard deviation $\beta$. Then, $\mu_t$, given $r_t$ has normal distribution with mean $\frac{\alpha^2 r_t^2}{\alpha^2 + \beta^2 r_t^2}$ and variance $\frac{\beta^2 r_t^2}{\alpha^2 + \beta^2 r_t^2}$. Here, assume that $\alpha=0.001$, $\beta=0.012$, $\sigma=0.025$. Then, the conditional distribution of $\mu_t$ given $r_t$ is normal with mean $0.81\alpha + 0.19r_t=0.19r_t+0.000813$ and standard deviation $0.01082$. Generating $r_t, t=1,2,...,n=100$ from normal distribution with mean 0.0023 and standard deviation 0.025, the histogram of the scaled first stopping time is given as follows.

Example 4 (CAPM modeling). In this example, a financial modeling of time varying $\mu_t$ is given. According to the CAPM theory the formula for calculating the expected return of an asset given its risk is as

$$\mu = r_f + \beta (r_{m} - r_f),$$

where $\mu$, $r_f$, $\beta$, and $r_{m} - r_f$ are expected return of investment, risk-free rate, beta of the investment and market risk premium, respectively. Investors expect to be compensated for risk and the time value of money. The risk-free rate in the CAPM formula accounts for the time value of money. The other components of the CAPM formula account for the investor taking on additional risk. The beta of a potential investment is a measure of how much risk the investment will add to a portfolio that looks like the market. If a stock is riskier than the market, it will have a beta greater than one. If a stock has a beta of less than one, the formula assumes it will reduce the risk of a portfolio. A stock’s beta is then multiplied by the market risk premium, which is the return expected from the market above the risk-free rate. The risk-free rate is then added to the product of the stock’s beta and the market risk premium. Other models for estimating the price of a financial asset such as three factor models of Fama and French is not considered, here. Interested readers may refer to Glen (2005) and references therein. Following French (2016), the time varying CAPM model is given by

$$\mu_t = r_f + \beta_t (r_{m} - r_f),$$

Here, the stock price of IBM and its market S&P500 is studied during 6 Feb. 2018 to 28 Feb. 2020 including 519 observations, taken from www.investing.com site. The risk free rate 0.014 is assumed. Then, the sequential slope of CAPM regression is computed. The following figure shows this series.
By finding $\mu$’s and computing $\gamma$’s, it is seen that the optimal stopping time occurs after 29 Jan 2020.

**CONCLUDING REMARKS**

Optimal stopping techniques, specially solved by dynamic programming in backward version, provide a useful framework for traders to decide when enter or exit of a financial position. Derived optimal stopping rules are applied to two real data sets which show the applicability of proposed technique and some other its features are presented in simulated illustrative examples.

**REFERENCES**