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New Sufficient Conditions for Hamiltonian and Traceable Graphs

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Abstract: In this note, we present new sufficient conditions for Hamiltonian and traceable graphs.

Keywords: Hamiltonian graph, traceable graph

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. For a graph G = (V, E), we use n and e to denote its order |V| and size |E|, respectively. The connectivity of the graph G is denoted by k(G). For disjoint subsets S, T of the vertex set V(G) of a graph G, let E(S, T) be the set of the edges in G that join a vertex in S and a vertex in T. We use G∨H to denote the join of two disjoint graphs G and H. The graph consists of p isolated vertices is denoted by Ep. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. In this note, we present new sufficient conditions for Hamiltonian and traceable graphs. The main results are as follows.

Theorem 1. Let G be a graph of order n ≥ 3, e edges, and connectivity k ≥ 2. If

$$e \geq (n - k - 1)(n + k)/2,$$

then G is Hamiltonian or $K_k \vee E_{k+1}$, where n = 2k + 1.

Theorem 2. Let G be a graph of order n ≥ 2, e edges, and connectivity k ≥ 1. If

$$e \geq (n - k - 2)(n + k + 1)/2,$$

then G is traceable or $K_k \vee E_{k+2}$, where n = 2k + 2.

Next, we will prove Theorem 1 and Theorem 2.

2 Proofs

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1. If G has a Hamiltonian cycle, then the proof is finished. Now we assume that G is not Hamiltonian. Choose a longest cycle C in G and give an orientation on C. Since G is not Hamiltonian, there exists a vertex x_0 in V(G) - V(C). By Menger's theorem, we can find s (s ≥ k) pairwise disjoint (except for x_0) paths $P_1, P_2, \ldots, P_s$ between x_0 and V(C). Let u_i be the end vertex of P_i on C, where 1 ≤ i ≤ s. We, without loss of generality, assume that the appearance of $u_1, u_2, \ldots, u_s$ on C agrees with the given orientation of C. We use u_i^+ to denote the successor of u_i along the given orientation of C, where 1 ≤ i ≤ s. Then a standard proof in Hamiltonian graph theory yields that $S := \{x_0, u_1^+, u_2^+, \ldots, u_k^+\}$ is independent (otherwise G would have cycles which are longer than C). Thus

$$\begin{aligned}(n - k - 1)(n + k)/2 &\leq e = |E(S, V - S)| + |E(G[V - S])|\\ &\leq |S|(n - |S|) + (n - |S|)(n - |S| - 1)/2\\ &= (n - |S|)(n + |S| - 1)/2 \leq (n - k - 1)(n + k)/2.\end{aligned}$$

Therefore |S| = k + 1, xy is in E for any vertex x in S and for any vertex y in V - S, and G[V - S)] is complete.

Let H be the component of G[V - V(C)] containing x_0. Since u_1^+ is adjacent to any vertex in V - S, H must consist of a singleton x_0 (otherwise G would have a cycle which is longer than C). Since x_0 is adjacent to any vertex in V - S, H must be the only component of G[V - V(C)] (otherwise x_0 would be adjacent to a vertex in another component of G[V - V(C)], which is a contradiction). Again, since x_0 is adjacent to any vertex in V - S, the segment from u_i^+ to u_{i+1} along the given orientation of C, for each i with $1 \leq i \leq s$ and s + 1 is regarded as 1, must consist of only u_i^+ and u_{i+1} (otherwise G would have a cycle which is longer than C). Hence G is

$$K_k \vee E_{k+1}, \text{ where } n = 2k + 1.$$

This completes the proof of Theorem 1. QED

Proof of Theorem 2. Let G be a graph satisfying the conditions in Theorem 2. If G has a Hamiltonian path, then the proof is finished. Now we assume that G is not traceable. Choose a longest path P in G and give an orientation on P. Let y and z be the two end vertices of P. We assume that the appearance of y and z on P agrees with the given orientation of P. Since G is not traceable, there exists a vertex $x_0$in V(G) - V(P). By Menger's theorem, we can find s ($s \geq k$) pairwise disjoint (except for x_0) paths $P_1, P_2, ..., P_s$ between x_0 and V(P). Let u_i be the end vertex of P_i on P, where $1 \leq i \leq s$. We, without loss of generality, assume that the appearance of $u_1, u_2, ..., u_s$ on P agrees with the given orientation of P. Since P is a longest path in G, $y \neq u_i$ and $z \neq u_i$, for each i with $1 \leq i \leq s$, otherwise G would have paths which are longer than P. We use u_i^+ to denote the successor of u_i along the given orientation of P, where $1 \leq i \leq s$. Then a standard proof in Hamiltonian graph theory yields that $S := \{x_0, y, u_1^+, u_2^+, ..., u_k^+\}$ is independent (otherwise G would have paths which are longer than P). Thus

$$\begin{aligned}(n - k - 2)(n + k + 1)/2 &\leq e = |E(S, V - S)| + |E(G[V - S])| \\ &\leq |S|(n - |S|) + (n - |S|)(n - |S| - 1)/2 \\ &= (n - |S|)(n + |S| - 1)/2 \leq (n - k - 2)(n + k + 1)/2.\end{aligned}$$

Therefore |S| = k + 2, xy in E for any vertex x in S and for any vertex y in V - S, and G[V - S)] is complete.

Let H be the component of G[V - V(P)] containing x_0. Since u_1^+ is adjacent to any vertex in V - S, H must consist of a singleton x_0 (otherwise G would have a path which is longer than P). Since x_0 is adjacent to any vertex in V - S, H must be the only component of G[V - V(P)](otherwise x_0 would be adjacent to a vertex in another component of G[V - V(P)], which is a contradiction). Again, since x_0 is adjacent to any vertex in V - S, the segment from u_i^+ to u_{i+1} along the given orientation of P, for each i with $1 \leq i \leq s-1$, must consist of only u_i^+ and u_{i+1} (otherwise G would have a path which is longer than P). Moreover, the segment from y to u_1 along the given orientation of P must consist of only y and u_1 and the segment from u_s^+ to z along the given orientation of P must consist of only u_s^+. Hence G is

$$K_k \vee E_{k+2}, \text{ where } n = 2k + 2.$$

This completes the proof of Theorem 2. QED

References

1. J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York (1976).

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