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Progressive Series: Development and Applications

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Abstract: Indian literature classifies progressive series into three categories viz. arithmetic, geometric and complex that classification came into effect in the middle of 9th century through the works of Mahāvīra. This paper is intended to present the work in the areas of arithmetic, geometric and complex series through geometrical, symbolical and algebraic interpretations from antiquity to seventeenth century in the Indian and other culture areas. Examples of significance, applications and comparative studies, apart from educational relevance, have been included.

Key Words: Arithmetic and Geometric Progressions; Complex Series; Series-Figure; rth Repeated Sum; Prime Numbers

I. INTRODUCTION

Arithmetic and geometric series or progressions are widely applied to various fields of applications including medicine (drug administration), physics (radioactive decay of nuclei), public policy and consumer affairs, finance, population dynamics, etc.

The terms progression and series are analogous in Hindu terminology. Vedic literature of about 2000BCE contains early examples of specific arithmetic and geometric series. The Sanskrit term for a series is *średhi* meaning literarily progression or any set or succession of distinct things or *śreni* meaning line, row, series etc. and the name for a term is *dhana* meaning any valued object. First term is called *ādi-dhana* (first term) and any other term *ista dharma*, desired *dharma*. When the series is finite, its last term is called *antya-dhana* and middle term the *madhya-dhana*. For simplicity of use these were later on used as *adi*, *ista*, *madhya* and *antya*. First term is also called *prabhava* (initial term), *mukha* (face) or its synonyms. The technical name for A.P. are *chāyā* or *pracāyā*, *uttara* (difference or excess), *vrddhi* (increment) etc.(see [2], [10]).

The common ratio in G.P. is technically called *guṇa* or *guṇika* (multiplier). Thus a geometric series specifically named *guṇa-śreḍhi*. The number of terms in a series is known as *pada* (step meaning number of steps in sequence) or *gaccha* (period). The sum is termed as *sarva-dhana* (total of all terms), *średhi-phala* (result of progression), etc. As reported in [2] the above technical terms occur commonly in almost all the Hindu treatises from *Bakhashāli Manuscript* [BM] (ca.200BCE) onwards.

1.1. Arithmetic Progression

The *Yajurveda* is possibly the earliest evidence of simple (finite) sequence as $1,3,5,...$ and $4,8,12,....,48.$

Taittriya Saṃhitā contains several examples: $1,3,5,...,19,29,.....,99$; $2,4,6,...,20$; $1,3,5,...,33$;

$4,8,12,...,$; $10,20,30,...$

Vājasneyi Saṃhitā contains even and odd series: $4,8,12,...,48$; $1,3,5,...,31$

In *Śulbasutras* of Āastamba and Baudhāyana, the emergence of series are due to lying out procedure of bricks for successive construction of fire *altar*. Āpastamba enunciates: "On (the occasion of) the first construction, (the altar-builder) should construct (the fire altar) knee with 1000 bricks; on the second, naval-deep with 2000 bricks; on the third, mouth-deep with 3000 bricks. (The number of bricks employed in constructing the fire altar becomes thus) greater and greater on each successive occasion. He who constructs to attain the Heaven, (should thus construct with) great, high and unlimited …. (number of bricks); so it is known from the ancient scriptures)" cf. [6, p. 217].

Wherein it means, the AP: 1000, 2000, 3000, ….

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Its reference goes back to the *Taittiriya Saṃhitā* and copiously in the *Satapatha Brahmaṇa*.

In *Bṛhaddevatā* (500- 400BC), the sum of the series 2+3+4+….+1000=500499 is given.

Datta [3, p.218] infers from the *Satapatha Brāhmaṇa* x.5.4.7 (Eggeling translation) the series

24+28+32+….+48 and its sum 252, and that from *Baudhāyana Śulbasūtra*

$$1+3+5+...+(2n+1)=(n+1)^2.$$

1.2. Geometric Construction

Pāṭigaṇita[PG] Rule 80-84of Śrīdhara (herein onwards referred to [10]).

The rule indeed is concerned with finding the partial areas (*phala*) of series-figure for successive cubits (*kara*) of altitude (*lambaka*). The primitive series-figure is basically a trapezium of altitude 1 unit (say in cubits) (gaccha) whose flank sides are equal, base is smaller than face, and resembles an earthen drinking glass (*śarāva*) in which width at base (*bhū* or *dharā*) is smaller than the face (*much* or *vaktra*).

The area of series-figure (*sreḍhiṣetra*) for first cubit of altitude is *a*, for second cubit of altitude is *a+d*, and so on. Thus, *a, a+d, a+2d,* ..stand for the successive terms of AP which are in fact areas of the series-figure or successive altitude *n* (*n*=1,2,…). *a* is called the first term (*ādi*) , *d* the common difference (*cāyā*) of the series and *n* number of terms (*pada* or *gaccha*).

According to construction, Fig.1a, by stretching out two extremities of base and face by threads.

$$base, b = a-\frac{d}{2}$$

$$face,\ f_n = a+\left(n-\frac{1}{2}\right)d,$$

where *n* stands for altitude.

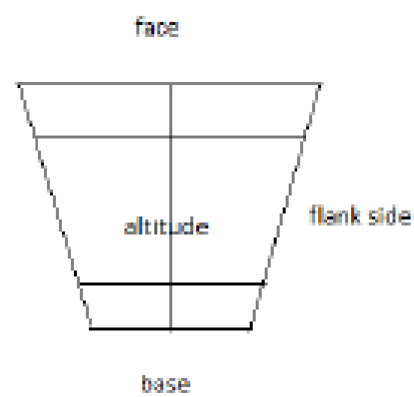

Figure 1a

For negative base, the series-figure, Fig.1b reduces to two triangles situated one over the other, by stretching the threads out crosswise.

The altitude of the upper triangle, $\dfrac{face}{face-base}=\dfrac{2a+d}{2d}$.

The altitude of the lower triangle, $\dfrac{face}{face-base}=\dfrac{d-2a}{2d}$.

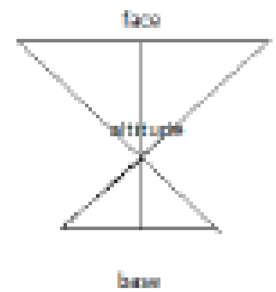

Figure 1b

1.3. Geometric Interpretation

PG Rule 85.

"The common difference as multiplied by one-half of the number of terms minus one, being increased by the first term, and then multiplied by the number of terms, gives the sum of the series.

And the area of the (corresponding) series-figure is equal to the product of one-half of the sum of the base and face, and the altitude".

That is, the sum of the series,

$$a+(a+d)+(a+2d)+....+n\,terms=\left\{\frac{n-1}{2}d+a\right\}n$$

and the area of the corresponding series-figure is equal to

$$\frac{base+\ face}{2}\times(altitude).$$

The BM seems to be the first to give a general rule for the summation of the series of the type

$$a+(a+d)+(a+2d)+....+[a+(n-1)d]$$

in the form

$$s=n[a+(n-1)\frac{d}{2}].$$

This rule of summation is reported in the works of *Āryabhaṭīya*[AB] of Āryabhaṭa I, *Brahmsphuṭasiddhānta*[BrSS] of Brahmagupta, *Mahāsiddhānta*[MS] of Āryabhaṭa II, *Gaṇitasārṅangraha*[GSS] of Mahāvīra, *Tristikā*[TS] of Śrīdhara, *Triloksāra*[TLS] of Nemicandra, *Siddhāntaśekhara*[SiS] of Śripati, *Līlāvatī*[LV] of Bhāskara II and *Gaṇitakaumudī*[GK] of Nārāyaṇa Paṇdita.

Remarkable here is that in agreement with the PG. Nārāyaṇa represents parallel sides of a trapezium by $a-\frac{l}{2}$ and $nl+a-\frac{l}{2}$ and its altitude by *n* (see Fig 1a). In case the sum comes out to be zero, the figure is the joint of two (equal) isosceles inverted triangles with their vertices [1, p. 186] (see Fig 1b).

All the above except Nemicandra even give the particular sum $1+2+3+...+n=\frac{n}{2}(n+1)$. The AB, PG, LV and GK give the result conforming to an AP.

PG Rule 86i. The first term is given by

$$a = \frac{s}{n} - \frac{d}{2}(n-1).$$

Also found in the GSS, MS, SiS, LV and GK [9, p.71; 10].

PG Rule 88.

$$a = \frac{\frac{n^2 - n}{2}(a+d) - s}{\left\{\frac{n-1}{2} - 1\right\} n}.$$

This rule is a manipulation of the Rule 85

$$s = \left\{\frac{n-1}{2}d + a\right\} n = \frac{n^2 - n}{2}(a+d) - \left(\frac{n-1}{2} - 1\right) na.$$

PG Rule 86ii.

$$d = \frac{\frac{s}{n} - a}{\frac{1}{2}(n-1)}.$$

The rule is also recorded in the TS, GSS, MS, SS and LV [10, p.70; 14].

PG Rule 87. The result

$$n = \left[\frac{\sqrt{8ds + (2a-d)^2} - 2a + d}{2d}\right],$$

taking positive sign of radical as n is positive.

Given primarily by the AB, various equivalent form forms the part of series rules of BrSS, TR, GSS, MS, SiS, LV and GK [10, p.71; 11]. Evidently, it is the root of quadratic equation

$$dn^2 + (2a-d)n - 2s = 0.$$

Śaṅkar in his *Kriyākramkarī* establishes the formula for n geometrically by means of rectangular piles using 8*sreḍhiṣetras* [26].The *LV* uses only 2 *sreḍhiṣetras.*

1.4. Symbolical Interpretation

PG Rule 89.

"The common difference (*cāyā*) as multiplied by the integral part of the number of terms (*nirvikalapada*) should be increased by the first term (*ādi*), and the result (obtained) should be kept undestroyed (at one place). The same result (written in another place) being increased by the first term (*mukha*), (then) diminished by the common difference, (then) multiplied by one-half of the integral part of the number of terms, and (then) added to the 'undestroyed result' as multiplied by the fractional part of the number of terms (*vikala*), gives the sum of the series (*gaṇita*)".

That is, for finding the sum of the series of the type

$$a + (a+d) + (a+2d) + ... + \left(n + \frac{p}{q}\right) terms,$$

where the number of terms breaks into integral and fractional parts, the former represents the sum of n terms and the latter $\left(\frac{p}{q}\right)^{th}$ part of the (n+1)th term, the Rule 89 corresponds to

$$s=\frac{n}{2}(dn+a+a-d)+\frac{p}{q}(dn+a).$$

PG Rule 90. If $n+\dfrac{p}{q}$ be the number of terms, then

$$a=\frac{s-\left\{\frac{1}{2}(n-1)+\frac{p}{q}\right\}dn}{n+\frac{p}{q}}.$$

It is a manipulation of Rule 89.

PG Rule91.

$$d=\frac{s-a\left(n+\frac{p}{q}\right)}{S},$$

where *S* denotes the sum of $\left(n+\frac{p}{q}-1\right)$terms of the series $1+2+3+....\quad i.e.\dfrac{(n-1)n}{2}+\dfrac{p}{q}n$ (vide Rule89).

PG Rule 92-93. If *n* denote the integral part of

$$\left[\frac{\sqrt{2ds+(a-d/2)^2}-(a-d/2)}{d}\right],$$

then the number of terms $\left(n+\frac{p}{q}\right)$of the series is equal to

$$n+\frac{s-\left\{(n-1)\frac{d}{2}+a\right\}n}{nd+a}.$$

For its rationale refer to [10, pp.77-82].

PG Rule 95ii. The multiplication of a number of bangles by half the sum of prices of the first and last bangles, gives the price (of all the bangles).

The rule evidently an application of the following formula,

$$sum\ of\ arithmetic series=\frac{first term+last term}{2}\times(number of\ terms).$$

PG Rule96: “Having subtracted the initial speed <*v* per day> from the constant speed <*u* per day>, divide the remainder as multiplied by 2 by the acceleration <*f* per day per day> in velocity. The quotient plus 1 gives the time elapsed <*n*>(in term of days) when the distances traversed are equal”.

That is,

$$n=\frac{2(u-v)}{f}+1.$$

This obviously follows from equating *un* with $\dfrac{n}{2}[2v+(n-1)f]$.

Cf. BM, GSS and GK [10, p.78]

PG Rule97-98. Rule for finding the number of days elapsed at the second meeting of two the two travellers since their first meeting.

"In relation to the first traveller, assume an arbitrary number, greater than the *cāyā* (i.e., acceleration) for the second traveller, for the *cāyā* (i.e. acceleration); and another arbitrary number (*ista*) for the *mukha* (i.e. initial speed). In relation to the second traveller, assume another arbitrary number to denote the *pada* (i.e., the number of days elapsed at the first meeting); and from the corresponding *pada* (i.e., the number of days elapsed at the first meeting) for the first traveller, calculate the *ādi* (i.e. the initial speed) for the second traveller.

Now divide the *phala* (i.e., the distance travelled by each traveller at their first meeting) severally by the *padas* (for the two travellers) and take the difference of the two; diminish that (difference) by half the difference of *cāyā* x *pada* for the two travellers; and then divide that by half the difference between the *cāyās* (for the two travellers): the quotient gives the days elapsed at the second meeting (of the two travellers) (since the first meeting)"

That is, the number of days (D) elapsed is given by

$$D - \frac{\left(\frac{s}{n-d} - \frac{s}{n}\right) - \frac{1}{2}\{nf_1 - (n-d)f_2\}}{\frac{1}{2}(f_1 - f_2)},$$

where f_1, f_2 are the accelerations of the two travellers, n is the number of days elapsed at the first meeting since the start of the second traveller, $(n-d)$ is the number of days elapsed at the first meeting since the start of second traveller, and s is the distance travelled by the two travellers at their first meeting. For its rationale refer to [10, p.80]. See also GK [10, p.79].

PG Rule 99-111. Rule to find the amount by which one gambler defeats his opponent in a play with a dice, when the moneys staked at the successive casts of dice are in an AP.

"Diminish the first *pada* (i.e., the number of casts of dice won in the beginning by either of the two persons) by one; (taking the remainder as the number of terms) find the sum of the series whose first term (*ādi*) and common difference (*cāyā*) are each one: this is the *vṛddhi* (for the first *pada*). In regard to the other *padas* (ie., the number of casts of dice won subsequently), take the sum of the preceding *padas* for the first term (*prabhava*), one for the common difference (and the *padas* for the number of terms, and find the sums of the series. These will give the *vṛddhis* for those *padas*).

Now taking the sum of all the *padas* (for the pada), find the um as in the case of the first *pada*; then diminish that (sum) by twice the *vṛddhis* corresponding to the *padas* of the lesser group (i.e. the *padas* corresponding to the person who wins lesser number of casts); then multiply (the remainder) by the (given) common difference; and then add that (product) to the product of the (given) first term (*ādi*) and the difference of the *padas* (of the greater and lesser groups): this gives the amount by which the person with greater number of *padas* (i.e., casts of dice in his favour) is victorious. If that quantity be negative, then it gives the amount by which the person with lesser number of *padas* is victorious".

Suppose that two persons A and B gamble with dice, and that day alternately win p_1, p_2, p_3 and p_4 casts. If the stake-moneys of casts are in the A.P. $a,\ a+d,\ a+2d, ...$ and $p_1 + p_2 > p_3 + p_4$, then the person with greater number of casts in his favour, i.e. A is victorious over B by the amount

$$a(p_1 + p_2 - p_3 - p_4)$$
$$+ d\left[\frac{1}{2}(p_1 + p_2 + p_3 + p_4 - 1)(p_1 + p_2 + p_3 + p_4) - 2\left\{\frac{p_2}{2}(2p_1 + p_2 - 1) + \frac{p_4}{2}(2(p_1 + p_2 + p_3) + p_4 - 1)\right\}\right].$$

For rationale refer to [10, p.81]

GK Rule1-2a Ch. III [15].

"Multiply 'the number of terms less 1' by the common difference. The (product) added to the first term is the last term. The last term added to the first term and then halved is the average (of all the terms). The average multiplied by the number of terms happens to be the sum (of the series in A.P.). Multiply half of 'the number of terms less 1' by the common difference. Add the product to the first term. (The sum) multiplied by the number of terms happens to be the sum (of the series in A.P.)".

That is, for the series $a, a+d, a+2d, \ldots,$ the nth (last) term $a_n = a+(n-1)d$.

If $n = 2m+1$ be odd, the middle term $M = a_{n-m} = \frac{n+1}{2} th\ term = \frac{a+a_n}{2}$.

If $n = 2m$ be even, the two middle terms $\frac{n}{2}$ th and $\frac{n+2}{2}$ th and their mean

$$M = \frac{a_{n-m} + a_{n-m+1}}{2} = \tfrac{n}{2} th\ term + \tfrac{1}{2}d = a + \left(\tfrac{n}{2} - 1\right)d + \tfrac{1}{2}d = \tfrac{a+a_n}{2}.$$

The sum $s = nM = \frac{n}{2}[2a+(n-1)d]$.

LV Ex 120. A person, having given four *dramas* to priests on the first day, proceeded, my friend, to distribute daily alms at a rate increasing by five a day. Say quickly how many were given by him in half a month.

LV Ex 124. On an expedition to seize his enemy's elephants, a king marched two *yojanas* the first day. Say, intelligent calculator, with what increasing rate of daily march he proceeded, he reaching his foe's city, a distance of eighty *yojanas*, in a week.

LV Ex 126. A person gave three *drammas* on the first day, and continued to distribute alms increasing by two (a day); and he thus bestowed on the priests three hundred and sixty *drammas*: say quickly in how many days.

GK Rule 5[15].

"The product (of the first term, the common difference and number of terms) divided by the desired first term is called *karma*. Add the *karma* to twice the first term. Multiply the sum by half of the *karma* added to the sum (of the series). (The quotient) is the common difference. Here, the karma divided by the common difference is the number of terms".

That is, if $k = nd$, then $d = \frac{k(2a+k)}{2s+k}$ and $n = \frac{k}{d}$.

Ex. The product of first term, the number of terms and the common difference is 12 and the sum of the series is 10. O friend, tell them (separately). What are the measures of the first term, the number of terms and the common difference, if their product and sum (of the series) be equal to one each.

GK Rule 6[15].

"One (progression) has greater first term and lesser common difference; the other has (lesser) first term and greater common difference, (the sum of the series and, their number of terms being equal) Divide the difference of their first terms by half of the difference of their common differences. (The quotient) added to 1 is the number of terms".

Let $a > A, d < D$. Accordingly,

$$\frac{n}{2}\left[2a+(n-1)d.\right] = \frac{n}{2}\left[2A+(n-1)D\right].$$

Hence the rule, $n = \frac{a-A}{\frac{1}{2}(D-d)} + 1$.

Ex. A traveller travels 9 *yojans* on the first day and 5 *yojans* more on each subsequent day. The other travels 2*yojans* on first day and 7 *yojans* on each subsequent day. Tell how many days and how much they travel (if they travel equally).

GK Rule7-8[15].

"The first term and the 'not lesser' common difference of the first (progression) and the number of terms of the second progression are desired. Supposing the first term (etc.) of the first (progression), obtain the sum (of the series in A.P.). Obtain the first term of the second, from the common sum. The first term added to the product of 'the number of terms and the common difference' happens to be r the first term for the second meeting (for each traveller). From those, the sum (of the series in A.P.) and the number of days (i.e., terms) should be obtained by the method stated earlier". -

Interpretation. Let a and n be the initial speed and the number of days for the first traveller A and N be those for the second traveller for the first meeting and a', n' and A', N' refer to those, in order, for the second meeting. Suppose that the first traveller starts with an increasing speed d per day and the second, that with D per day. Also let the second traveller start when the first traveller has move for t days and, $d \geq D.$ D and t are given. According to the rule a, d and N are supposed. Now, $n = t + N$ and, s and A are obtained by the methods stated earlier and considering the fact that the distance traversed by the two travellers for the first meeting is the same.

For the second meeting, according to the rule $a' = a + nd$ and $A' = A + ND.$ clearly a' and A' are the terms next to the last term for the first meeting. Rule6 gives the time $\frac{2(A'-a')}{d-D}+1$ between two meetings and so, $N' = \frac{2(A'-a')}{d-D}+1+N$ and $n' = t + N'$ and therefore the distance covered before the second meeting can be found out.

According to Nārāyaṇa, in case the time obtained between two meetings is negative (-t), $N'(= N - t)$ and $n'(= n - t)$ give the number of days for the first meeting and$N$ and n, those for the second meeting.

GK Rule 19b-20a[15]. Repeated sum *Varasamkalita* of rth order of a natural number.

"Number of terms (say n, is) the first term (of an A.P.) and 1, the common difference. Those (i.e. terms of A.P., their being) 1 more than the number of times (the sum is to be taken, i.e., r+1), separately, (are) the numerators. 1 (is) the first term (of another AP) and 1, the common difference. These are the denominators, (their number being the same as that of the former A.P.). Their product (is) the rth<order repeated>sum of n".

That is, rth order repeated sum of n,

$$^{n}S_r = \sum\sum.....\sum n = \frac{n(n+1)(n+2)+...+(n+r)}{1.2.3......(r+1)} = {}^{n+r}c_{r+1}.$$

[(The number of $\sum s$ being r]

GK Rule 20b-21Ch. III [15].The rth repeated sum of a series in A.P.

"The rth <repeated> sum of 'number of terms less 1' is multiplier (of the common difference). That separately multiplied by '1 more than the number of times' (the sum is to be taken, i.e.,r+1 and then) divided by 'the number of terms less 1' is multiplier of the first term. The first term and the common difference (are both) multiplied by their own multipliers. The sum of the products happens to be the rth sum (of the series in A.P.)".

Let an AP be $a, a+d, a+2d,...., a+(n-1)d.$ Then rth repeated sum of AP will be

$$= \left[\frac{a(r+1)}{n-1}+d\right]\frac{(n-1)n(n+1)...(n+r-1)}{1.2.3.....(r+1)} = \left[\frac{a(r+1)}{n-1}+d\right]{}^{n-1}S_r$$

For its rationale, see [15].

Ex GK. First term (of an AP) is 5, common difference, 3 (and) the number of terms, 7. O best among learned, tell quickly the 4th<repeated> sum (of the series in AP). (Also), if you have passion or mathematics, tell the sum by changing (the ingredients).

G Rule 20b-21 Ch. III [15]. Cow Problem

Subtract the number of years <*k*> (in which a calf begins giving birth) from the (total) number of years <*n*> (successively and) separately, till the remainder becomes smaller (than the subtractive). Those are the numbers for repeated summations once, (twice) etc. in order. Sum of the summations along with 1 added to the total number of years is the number of progency.

That is, corresponding to an A.P. $n-k, n-2k, ..., n-rk$, where $n-rk<k,\ i.e.\ r>\frac{n-k}{k}.$

Number of progency, $N={}^{n-k}S_1+{}^{n-2k}S_2+{}^{n-3k}S_3+......+{}^{n-rk}S_r+1+n.$

Ex. GK. A cow give birth to a (she) calf every year (and) their calves themselves begin giving birth) when 3 years old. O learned, tell the number of progency by a cow during 20 years.

Putting $r=1,2,.....$, we find

$${}^nS_1=\sum_{m=1}^{n} m=\frac{n(n+1)}{1.2}$$

$${}^nS_k=\sum_{m=1}^{n} {}^mS_{k-1}=\frac{n(n+1)....(n+k)}{1.2.....(k+1)},\ k>1$$

Theorem. If ${}^nV_1=\sum_{m=1}^{n}\alpha_m$, where $\alpha_1,\alpha_2,....,\alpha_m$ be an A.P. with α_1 the first term and β the common difference, then ${}^nV_k=\sum_{m=1}^{n} {}^mS_{k-1}$ and ${}^nV_k=\alpha_1\frac{k+1}{n-1}{}^{n-1}S_k+\beta^{n-1}S_k.$

II. Geometric Progression

The *Pañcaviṃsa Brāhmaṇa* contains the sequence $12,24,48,....,196608,393216$ while the Buddhist work *Dīgha Nikāya* is claimed to have used $10,20,40,....,80000$ in connection with increment in human life.

Noticeable here is that Noticeable that *Chandah-sūtra* (rules of meters) of Piṅgla mentions variations corresponding to *n* syllables which comes out in the form $1,2^1,2^2,....,2^n.$

The Jana work of Bhadrabāhu (ca. 300BCE) (*Kalpa Sūtra*) mentions 1+2+4+…+8192=16383.

Commentary Dhavalā of Vīrasena (9th century AD) on Ṣatkhaṇḍāgama of Puṣhpadanta Bhūtabali

$$49\frac{217}{452}\left(1+\frac{1}{4}+\frac{1}{4^2}+....\right)=65\frac{110}{113}.$$

GK Rule 17b-18a (Ch XIII)[19].

"Write 1 in the beginning and then that (i.e., 1) multiplied by the greatest digit, ahead of that. Again, in the same way, (the numbers multiplied by the greatest digit written ahead) until (the number of terms is) 1 more than the number of places is called the Geometric Progression".

Let q be the greatest digit and p, the number of places. Then G.P. will be $1,q,q^2,....,q^p.$

For a given geometric series to *n* terms $a,ar,ar^2,....,ar^{n-1}$, how to obtain (n+1)th term, i.e. $ar^n.$

PG Rule 94.

"When the number of terms of the series (i.e., the number denoting the period) is odd. Subtract 1 from it and write 'multiply (by the common ratio)'; and when the number of terms of the series is even, halve it and write 'square'. (Apply the same rule to the resulting number, and continue the process till the number reduces to zero). Having thus written down 'multiply' and 'square' in a sequence (write 1 thereafter). Then starting from 1bachwards, perform the operations of multiplication and squaring, and finally multiply the resulting quantity by the first term of the series (i.e. the given sum)".

PG Rule 95i."The result obtained according to the previous rule, being diminished by the first term of the series, and (then) divided by the common ratio minus 1, gives the sum of the sum of the series".

Whence,

$$a + ar + ar^2 + ... + n\,terms = \frac{ar^n - a}{r-1},\ r > 1$$

The above rule in accumulated form exist in the LV, GSS, MS and GK. The SiS too contains in it the Rule 94 whereas Pṛthudakasvāmi and Nemicandra on other hand give the result of sum, i.e. Rule 95i.

Ex PG. Some businessman, taking 3 *rupās* with him, went out to make profit. If his capital becomes double after every month, what will it become after 3 years?

Here $a = 3, r = 2, n = 36$

Denote S for squaring at even number and M for multiplying at odd ones

Proceeding as per rule:

Number of Terms	**Value**	**Operations**	**Write**
36(even)	$68719476736 = 2^{36}$	36/2=18	S
18(even)	$262144 = 2^{18}$	18/2=9	S
9(odd)	$512 = 2^{9}$	9-1=8	M
8(even)	$256 = 2^{8}$	8/2=4	S
4(even)	$16 = 2^{4}$	4/2=2	S
2(even)	$4 = 2^{2}$	2/2=1	S
1(odd)	$2 = 2^{1}$	1-1=0	M

Proceeding from the last (i.e. 1) backwards and performing the operations of multiplication (by the common ratio 2) and squaring successively, as indicated in the last column, we obtain $2, 2^2, 2^4, 2^8, 2^9, 2^{18}, 2^{36}$ and sum $3x2^{36} - 3 = 206158430205$.

LV Ex 128.The initial quantity being two, my friend; the daily augmentation, a threefold increase; and the period, seven; say what the sum in this case is.

Proceeding as per rule:

Number of Terms	Value	Operations	Write
7(odd)	$2187 = 3^{7}$	7-1=6	M
6(even)	$729 = 3^{6}$	6/2=3	S
3(odd)	$27 = 3^{3}$	3-1=2	M
2(even)	$9 = 3^{2}$	2/2=1	S
1(odd)	$3 = 3^{1}$	1-1=0	M

Sum= $2\left(2^7 - 1\right)/2 = 2186$.

LV Ex 129. The initial quantity being two, my friend; the daily augmentation, a threefold increase; and the period, seven; say what the sum in this case is.

GK Rule 25Ch III[15]."Divide the sum by the first term. Subtract 1 (from the quotient). Divide (the difference) by a factor which leaves no remainder. 'Subtract 1 from the result (and then) divide by the same factor', again and again, until 1 is obtained. The divisor is the multiplier (i.e.) the common ratio.

That is, for finding the common ratio *r*, one has to search the identity

$$\frac{1}{r}\left(\frac{1}{r}\left(....\frac{1}{r}\left(\frac{1}{r}\left(\frac{s}{a}-1\right)-1\right)-1\right)-1......1.\right)=1.$$

GK Rule26-27a[15].

"Divide the sum by the first term. Multiply (the quotient) by the common ratio less1. Add 1 (to the product). Divide (the result)) by the multiplier (i.e., by the common ratio) again and again till the end, (for each division) keep unity. The sum of that (i.e., of unities) is the number of terms".

Accordingly, divide $(r-1)\left(\frac{s}{a}\right)+1$ by *r* again till the end. The number of division gives *n*.

Mahāvīra also gives deduction rules for finding *a*, *r* and *n* of a geometric series.

2.1. Geometric Representation (Series-Figure)

GK Rule64b-65a[17].

"Divide the face by base. (Face and base etc.) multiplied by (the quotient) are the face etc. for the quadrilateral standing above that (quadrilateral). Divide the base by the face. Face, (base) etc. multiplied by (the quotient) happen to be the face etc. in the quadrilateral lying below (that quadrilateral)".

In such quadrilaterals, lengths of their respective flank sides, bases, faces, respective diagonals, respective perpendicular and areas are a series in G.P.

Let *b* be the base and *f* the face of the given quadrilateral. The common ratio for constituents of former set (except the area) will be *f/b* and that of latter set *b/f*. The common ratio for the area will be f^2/b^2 in the former case and b^2/f^2 in the latter case.

Fig.2 shows graphic demonstration of $s=\frac{ar^n-a}{r-1}$, $r>1$ (see [19], [23], [25). The *Kriyākramkarī* case for $r=4$ can be generalized for any r. The long rectangular strip ABCD represents the (n+1)th term (i.e. T_{n+1})is divided into three (i.e. r-1) equal parts of which AEFD into r (i.e. 4) equal parts. Thence first three (.e. r-1) divisions comprising of strip AGHD will be

$$\frac{3}{4}\frac{1}{3}T_{n+1}=\frac{1}{r}T_{n+1}=T_n$$

Repeating the same process for strip GEFH as shown, GIJH corresponds to T_{n-1}. This process of division is repeated till T_1. Hence

$$\frac{1}{3}T_{n+1}=\text{Rectangle AEFD}=T_n+T_{n-1}+.....+T_1+\frac{1}{3}T_1=s+\frac{1}{3}T_1, \text{ i.e.}$$

$$\frac{1}{r-1}ar^n=s+\frac{1}{r-1}a.$$

Hence the rule follows.

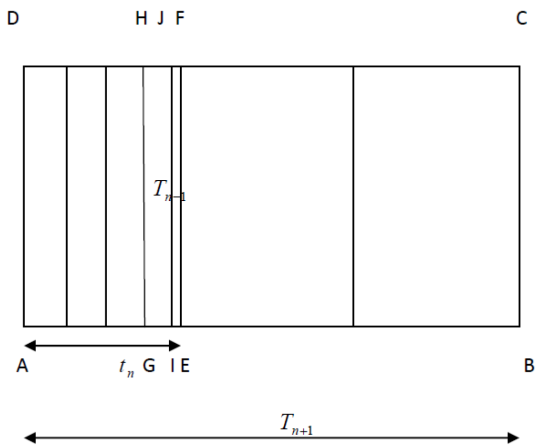

Figure 2

2.2. Complex (Special) Series

I. LV, AB, BrSS, SiS and PG

$$1+(1+2)+(1+2+3)+....+(1+2+3+...+n\,terms)$$
$$=\frac{n(n+1)(n+2)}{6}$$
$$=\frac{(n+1)^3-(n+1)}{6}.$$

II. LV, AB, BrSS, SiS, GSS, PG and GK

$$1^2+2^2+.......+n^2=\frac{n(n+1)(2n+1)}{6}$$

III. AB, BrSS, SiS, GSS, PG, LV, GK

$$1^3+2^3+.......+n^3=\left[\frac{n(n+1)}{2}\right]^2$$

IV. AB, BrSS, SiS, GSS, LV and GK

$$\sum\frac{r(r+1)}{2}=\frac{(n+2)(n+1)n}{6}$$

V. PG, GSS and GK

$$\sum_{m=1}^{n}\sum_{r=1}^{m}\frac{r(r+1)}{2}+n^2+n^3=\frac{(2n+1)(n+1)n}{2}.$$

VI. PG

$$\sum_{m=1}^{n}\sum_{r=1}^{m}\frac{r(r+1)}{2}+\sum n^2+\sum n^3=\frac{n(n+1)^2(n+2)}{4}.$$

The Series of squares and cubes corresponding to $a, a+d, a+2d,...., a+(n-1)d$:

a) $\sum_{r=1}^{n}\{a+(r-1)d\}^2 = [a+(a+2d)+(a+4d)+\ldots to\,n\,terms]a+\left[1^2+2^2+\ldots.+(n-1)^2\right]d^2$

[PG 105]

$= a\left[\frac{n}{2}(2a+2n-2d)\right]+d^2\left[1^2+2^2+\ldots.+(n-1)^2\right]$ [GK 176-180]

$= n\left[\left\{\frac{(2n-1)d^2}{d}+ad\right\}(n-1)+a^2\right]$[GSS cf. 1, p. 183]

b) $(1+2+\ldots+a\,terms)+(1+2+\ldots+a+d\,terms)+(1+2+\ldots+a+2d\,terms)+\ldots to\,n\,terms$

$= \sum_{r=1}^{n}\frac{\{a+(r-1)d\}\{a+(r-1)d+1\}}{2}$

$= \frac{1}{2}\left[\sum_{r=1}^{n}\{a+(r-1)d\}^2+\sum_{r=1}^{n}\{a+(r-1)d\}\right]$

[PG106, GSS cf. 9, p.85]

$= \frac{n(n+1)}{2}\left[\frac{(a+d)(a+d+1)}{2}-\frac{a(a+1)}{2}\right]+\frac{na(a+1)}{2}+\frac{(n-1)(n-2)d^2}{1.2.3}$ [GK 16-17a]

c) $\sum_{r=1}^{n}\{a+(r-1)d\}^3 = s^2d+as(a-d)$ [PG 107, GK 18b-19a]

$= \begin{cases} s^2d+sa(a-d)\,if\ a>d \\ s^2d-sa(a-d)\,if\ a<d \end{cases}$ [GSS cf. Bag, p.183]

The BM notices two types of elementary complex series viz. *yutivargakrama* and *yutagaṇita-yutakarma* without summation. Mahāvīra (cf. [1], p.186) was successful in this aspect by giving the sum.*ī*

$$a+(ar\pm m)+[(ar\pm m)r\pm m]+[\{(ar\pm m)r\pm m\}r\pm m]+\ldots.to\,n\,terms = s'\pm\frac{\left(\frac{s'}{a}-n\right)m}{r-1},$$

where $s' = a+ar+ar^2+\ldots.to\,n\,terms.$

Ex GK III.13. O friend, if you are an expert in mathematics, tell the sum of (a series of) sums of first natural numbers, that of (a series of) squares and the same of (a series of) cubes, the first term, the common difference and the number of terms of the progression being 3, 4 (and) 9, respectively.

The *Āryabhaṭiyabhāṣya*, AB commentary on Āryabhaṭa I, of Nīlakaṇṭha Somayaji or Somasutvan of Kerala, presents some interesting geometrical derivations of series [24]. Accordingly, for example, $2s_n$, where $s_n = 1+2+3+\ldots..+n,$ is the area of rectangular strip whose length is $n+1$ and breadth *n* formed by joining two identical series-figures representing $1+2+3+\ldots..+n$ after inverting one of them. That is, $2s_n = n(n+1)$. s_n can also be represented as rectangular slab of length *n*+1, breadth *n*/2 and thickness 1(see Fig.3& Fig.4).

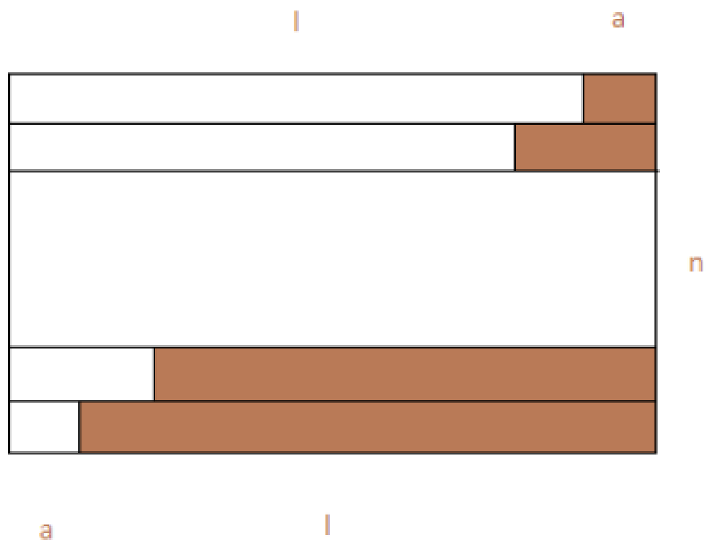

Fig3.

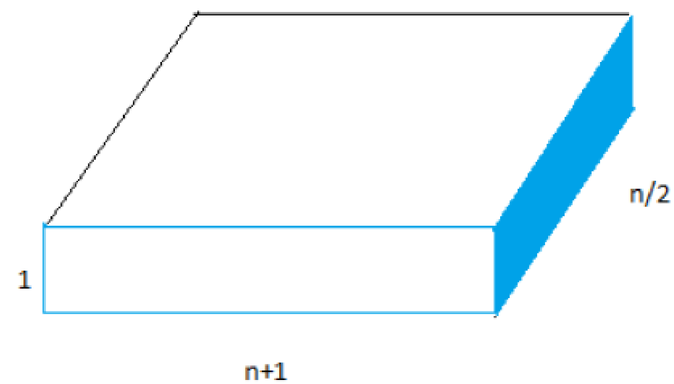

Fig4.

Using six rectangular slabs for each representing $s_n, s_{n-1}.....s_1$ generated of sides $n, n+1$ and $n+2$. Following exposition is easily established:

$$6s_n + 6s_{n-1} + ... + 6s_1 = n(n+1)(n+2),$$ i.e.,

$$s,+s_{n-1} + ... + s_1 = \frac{n(n+1)(n+2)}{6}.$$

Similarly other demonstrations for series for II and III exist in the AB commentary [25].

III. Miscellaneous Applications

a) GK Rule1, Ch.XII [17]

"Multiply the consecutive terms of the arithmetic progression whose first term and common difference are 1 each, two at a time. (The products) are the denominators, the last term (of the series) being the last denominator. (The fractions) when added (yield) 1 (as the sum). For any other desired sum, every fraction is to be multiplied by that number".

The rule implies

$$1=\frac{1}{1.2}+\frac{1}{1.2}+......+\frac{1}{(n-1)n}+\frac{1}{n}.$$

For any number *a*,

$$a=\frac{a}{1.2}+\frac{a}{1.2}+......+\frac{a}{(n-1)n}+\frac{a}{n}.$$

b) GK Rule1, Ch.XII [16].

"Terms of the Geometric Progression (whose) first term is 1 and, common ratio is 3, up to the number of places and the first term and the last term, multiplied by 2, are the denominators. The numerator of the last is multiplied by 3. (The fractions) when added become 1".

$$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{3^2}+......+\frac{1}{3^{n-2}}+\frac{3}{2.3^{n-1}}.$$

c) GK Rule 58 Ch XIII [19]. Combination n digits taking r of them at a time $C(n,r)$.

"(Number of) the arithmetic progression having the same number of places as are the digits (or things) are the divisors (and the same), in the reverse order, are the dividends. Multiply successively, the succeeding one by preceding (quotient). (The products) determine the number of combinations taking 1 etc. at a time". That is

$$\frac{n}{1},\frac{n(n-1)}{1.2},\frac{n(n-1)(n-3)}{1.2.3},......,\frac{n(n-1)....2.1}{1.2......(n-1)n}$$

are the combinations of n different digits taking $1,2,.....(n-1),n$ of them at a time, in order.

The relation $\sum_{r=0}^{n}C(n,r)=2^n$ is well known in Indian mathematics [27], in which

$$C(n,r)=\frac{n(n-1)......(n-r+1)}{1.2....r},r=1,2,..n.$$

d) GK Rule 62-63a Ch XIII [19]. Permutations of q digits taking p of them at a time $P(p,q)$.

"The last number of the geometric progression is the (total) number of permutations. The sequence o augmented units is multiplied by that (i.e. by the last number of geometric progression). Except the last, the remaining (products), divided by the greatest digit, are the number of permutations ending in 1,2, (etc.), in the reverse order. The sum of the digits in a particular place in all permutations etc. should be obtained by the method stated earlier".

Let the digits from 1 up to *q* take part in permutations and any number repeated in any permutation be less than or equal to the number of digits *p*. Herein *p* is called the number of places and *q*, the greatest digit.

The rule may be summarized as:

The number of permutations $=q^p$.

The number of permutations ending in $q,q-1,......2,1=q^{p-1}$.

The sum of digits at a particular place in all permutations is

$$(1+2+......+q)q^{p-1}=\frac{q(q+1)}{2}q^{p-1}.$$

The sum of all permutations is equal to

$$\frac{q(q+1)}{2}q^{p-1}\left[10^{p-1}+10^{p-2}+.......+10^2+10+1\right].$$

Total number of digits $= pq^p$.

For rationale see [19].

e) GK Rule 65-66 Ch XIII [19]

"Taking 'the greatest digit less 1' as the common ratio, write the geometric progression in the reverse order and 'the sequence of polynomial coefficients when the greatest digit is 2', below that. Multiply (the numbers of) the sequence of polynomial coefficients by the corresponding numbers above these, severally. These (i.e. the products) happen to be the number of permutations without 1, containing 1, once, twice, thrice, etc. or without 2, containing 3, once, twice, thrice, etc".

Writing G.P. in reverse order:

$$(q-1)^p,(q-1)^{p-1},......,(q-1)^2,(q-1),1$$

The $(r+1)$ th term will be $(q-1)^{p-r}$. For $q=2$

$$P(p,q,r)=C(q,r), r=0,1,2,...., p.$$

According to the rule, number of permutations in which a digit occurs r times $C(p,r)(q-1)^{p-r}$.

f) GK XI Rule 6b-7a[165].

"In case of a non-square number, (if) twice the square-root of its nearest square increased by 1 (and then) lessened by the difference (of the non-square number and the square-root) be a square, it is to be taken as the number to be added to (the non-square number to) make it a perfect square".

Let $N=a^2+r$ and $2a+1-r=b^2$. Then $N=(a+b+1)(a-b+1)$.

This rule is even reported in work of Śripati.

The Rule XI 7b-8a below is extension of above rule [op. cit].

"In case the above number is not a perfect square, it should be further increased (by the successive terms of the arithmetic series whose) first term is (the sum of) twice the nearest square-root (of the non-square number) added to 3, (and whose) common difference is 2, until it becomes a perfect square".

Rationale. Suppose that $2a+1-r$ is not a perfect square. The r terms of the series $2a+3,2a+5,....,2a+2r+1$ if added to $2a+1-r$ will make it a perfect square $(=k^2)$.

$$\begin{aligned}N+k^2 &= a^2+(r+2a+1)-r+(2a+3)+(2a+5)+....+(2a+2r+1)\\ &= a^2+(2a+1)+(2a+3)+(2a+5)+....+(2a+2r+1)\\ &= a^2+2a(r+1)+\frac{r+1}{2}[2+(r+1-1)2]\\ &= a^2+2a(r+1)+(r+1)^2\\ &= (a+r+1)^2.\end{aligned}$$

Thus $N=(a+r+1)^2-k^2=(a+r+k+1)(a+r-k+1)$

The above rules were rediscovered by French mathematician Pierre de Fermat (1601-1665).

A.P. in Magic Square

Let *n* be the total number of cells in the magic square (MS) and N its order. The MS is called double even, single even or single odd accordingly $n = 4m+2, 4m+1 or 4m+3.$ Notice that no. of terms in A.P.= Total no. of cells in MS = *n*.

Further, square constant *S* is defined by

$$S = \frac{T}{N} = \frac{n\left(2a+(n-1)d\right)}{2N}.$$

No. of steps = N= Number of cells in a row (*caraṇa*) of MS = Order of MS.

For $a=1, d=1,\ T = \frac{n}{2}\left[2a+(n-1)d\right] = na + \frac{n(n-1)}{2}d = na + sd,$ where sum of all natural numbers $s = \frac{n(n-1)}{2}.$

This means the first term *a* and common difference *d* of the A.P. are to be solved using pulverizer (*kuṭṭaka*) for generating equation $-sd + T = na.$ For details refer to [18].

Ex GK Ch. XIV. O learned, if you have pride in mathematics, tell the integral first term and common difference of magic squares (whose) total number of cells is as stated earlier (i.e. 16, 36 or 9) and (whose) totals are 400, 1296 and 180, in order.

g) Triangles with Sides of Prime Numbers

Brahmagupta analysed the class of so-called Heronian triangles having consecutive integer sides, popularly called the Brahmagupta triangles. Brahmagupta denoted the first eight such triangles as the triples (3,4,5) , (13,14,15), (51,52,53), (193,194,195), (723,724,7251), (2701,2702,2703), (10083,10084,10085) and (37633,37634,37635). Beauregard and Suryanarayan[20] found the matrix based iterative solution for a second degree indeterminate analogous equation.

IV. CONTRIBUTIONS IN OTHER CULTURE AREAS

The formula III, however, appeared I the work *Codex Arcerianus* (6th century), the proof of which and also of II was furnished by the Arabic *al-Karkhi* (ca. 1020) [1, p.183].Precious Mirror of Chu Shih-chieh (fl. 1202-1261) contains II without proof [5, p. 227].

Pythagoreans thoroughness for figurate numbers [5, p. 59-60].

The sequence of numbers $1+2+3+...+n = \frac{n(n+1)}{2}$ is called figurate numbers. The sequence of odd numbers $1+3+5+...+(2n-1) = n^2$ is termed as gnomon, resembling Babylonian shadow clock. The sequence of even numbers $2+4+6+...+2n = n(n+1)$ is called oblong numbers, double the triangular numbers.

The sequence $1+5+9+...+(4n-3) = 2n^2 - n$ is called the hexagonal numbers. The process continuously extends to polyhedral numbers.

The sum $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+.... = 1$ can be visually represented as a big square whose side's length 1 (see Fig. 5), is divided into sequence of triangles whose lengths are subsequently reduced to half of preceding length. Aristotle in *Physics* describes argument of Zeno of Elea (490-430 BCE) as "There is no motion because that which is moved must arrive at the middle of its course before it arrives at the end". It was an attempt to sum $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....$ backwards [23, p. 48].

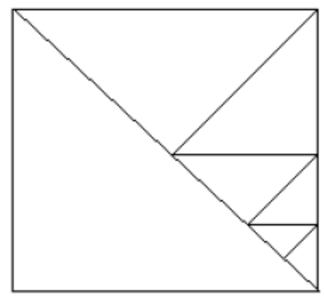

Fig5

Bagni[8] traces Aristotle for the first notion of infinite series according to sum of series of infinitely many potential addends that can be a finite quantity. According to Guido Grandi (1671-1742)

$$1-x+x^2-x^3+....=\frac{1}{x+1}. \tag{1}$$

This is valid if $|x|<1.$ Grandi, Leonhard Euler(1707-1783) and Fourier thought that $1-1+1-1+....=\frac{1}{2}.$ But it was Gottfield Wilhelm Leibniz (1646-1716) who prove that this statement is absurd it should either be 0 or 1 depending on number of terms taken even or odd.

Euler committed yet another mistake by concluding $1+2+4+8+....=-1$ from $1+x+x^2+x^3+....=\frac{1}{1-x}$ by putting $x=2$ [7]. This represents area under the hyperbola $y=\frac{1}{1+x}.$

Rhynd Papyrus (2000-1800BCE), named after Scottish Egyptologist A Henry Rhind, is one of the oldest Egyptian treatise on mathematics. Ahmes (1680-1620BCE), an official Scribe of Rhind Papyrus copiously derived it from a prototype dating back to 2000-1800BC.

It contains an example: “There are seven houses; in each house there are 7 cats; each cat kills seven mice; each mouse has eaten 7 grains of barley; each grain would have produced 7 hekat. What is the sum of all the enumerated things?”

The concept of geometric series began with Egyptians. The idea was conceived and developed by Greeks Thales (624-547 BCE), Eudoxus (400-350 BCE) and Pythagoras (ca. 580 BCE) based on theory of proportions where, for example, method of exhaustion was used to exhaust a circle cutting it up into square and triangles. Archimedes (287-212 BCE) of Greek successively used the method of exhaustion for confirming volume and surface area of sphere and area of parabolic segment. Surprisingly, for latter, in the area of parabolic segment the geometric series $1+\frac{1}{4}+\frac{1}{4^2}+....=\frac{4}{3}$ have been used. The area achieved by cutting off parabola by a chord perpendicular to the axis of symmetry and further dividing the parabolic segment into triangles [9, p. 44-46].

In 1930, Ncole Oresme gave a more general form of geometric series

$$\frac{a}{k}-\frac{a}{k}\left(1-\frac{1}{k}\right)+\frac{a}{k}\left(1-\frac{1}{k}\right)^2+....=a,$$

where $k>0$ stands for real set of numbers.

John Wallis (1616-1703) used geometric series for finding area under a hyperbolic curve. Newton used geometric series to divide polynomials. Integrating the geometric series (1) we find

$$A(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} -,$$

where $A(1+x)$ is natural logarithm of $1+x$.

This power series appeared in the work of Nicole Mercator (c.1668) [9, p. 120].

The pairing of terms of an A.P. with G.P. corresponding was the idea behind John Napier (1550-1617) to realise the Napier's computations of Logarithms. Later on Jobst Burgi (1522-1632) continued the similar idea of Napier but the methodology was based on algebraic rather than geometric [27].

Proposition.35 Book IX (Euclid) devices a formula equivalent to [5, p. 127]

$$s_n = \tfrac{a-ar^n}{1-r},\ r < 1.$$

Later on Fermat used this sum to establish area under a curve $y = x^n$ [5, p.385].

Further last Proposition of this Book states in modern equivalent that if $s_n = 1+2+2^2+....+2^n = 2^{n-1}$ is prime, then $2^{n-1}(2^n - 1)$ is a perfect square [5, p. 128].

In 1350, Oresme summed the series $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{3^3} + ... = 2$ by a geometric method most likely by cutting up an area of two square units. [9, p. 119].

However, first non-geometric example appeared in England in *Liber calculationum* by the logician Richard Suiseth (fl. ca. 1350), popularly known as Calculator. This statement is equivalent to: "If throughout the first half of a given time interval a variation continues at a certain intensity, throughout the next quarter of the interval at double this intensity, throughout the following eighth at triple the intensity and so ad infinitum; then the average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the initial intensity)".He gave a long and tedious verbal proof [5, p.293]

Egyptians are known to have used A.P. around 1550 BCE. The series $\sum n$ and $\sum n^2$ known to Pythagoreans and Archimedes made use of them in the quadrature of cubes. Nicomachus gave an expression for $\sum n^3$ which subsequently appeared in the arithmetic of Boethius. Arabs (Ibn-al-Haitham) made use of $\sum n^3$ and $\sum n^4$ in finding the volumes of solids of revolution. Stevin (sixteenth century) was the first to provide an arithmetic method based on the power series $\sum n^2$. In the seventeenth century Falhaber, visited at Ulm by Descartes in 1620, found expressions for the sums of the powers of the natural numbers up to 13[14, p. 151].

There has been development of arithmetic series of prime numbers. The simplest A.P. of five primes 5, 11, 17, 23, 29 are all known. In 2004, Frind, Jobling and Underwood [4] produced an example of 23 primes in A.P. using $6211383760397 + 44546738095860k,\ k = 0,1,2,....,22.$ For further details refer to [4].

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